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SHORTER COMMUNICATIONS

DEVIATIONS FROM CLASSICAL FREE CONVECTION BOUNDARY-LAYER THEORY AT LOW PRANDTL NUMBERS

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NOMENCLATURE

- c_p, specific heat at constant pressure;
- F, function of η, equations (12a, 13a, 14a, 15);
- f_i, functions of η;
- G, function of η, equations (12b, 13b, 14b, 15);
- Gr_x, Grashof number, $g\beta(T_w - T_\infty)x^3/\nu^2$;
- Gr_L, Grashof number, $g\beta(T_w - T_\infty)L^3/\nu^2$;
- g, acceleration of gravity;
- k, thermal conductivity;
- Pr, Prandtl number;
- p, static pressure;
- q, local heat flux at wall;
- T, temperature;
- u, streamwise velocity component;
- v, transverse velocity component;
- x, longitudinal coordinate;
- y, transverse coordinate.

Greek symbols

- β, coefficient of thermal expansion;
- η, similarity variable, equation (8);
- θ, dimensionless temperature, $(T - T_\infty)/(T_w - T_\infty)$;
- θ_i, functions of η;
- μ, viscosity;
- ν, kinematic viscosity;
- ξ, perturbation parameter, $Gr_x^{-\frac{1}{4}}$;
- ρ, density;
- ψ, stream function.

Subscripts

- 0, in the absence of $\partial p/\partial y$ and $\partial^2/\partial x^2$;
- w, at the wall;
- ∞, in the ambient fluid.

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INTRODUCTION

In conventional boundary-layer theory, the neglect of transverse pressure variations and streamwise second derivatives is based on the boundary-layer thickness being small (relative to the distance from the leading edge). For the case of low Prandtl number free convection on a vertical plate, relatively large boundary-layer thicknesses are encountered. For instance, for $Pr = 0.003$ (liquid sodium), the boundary-layer thickness is approximately 14·5 times that for $Pr = 0.733$ (air)*. Therefore, for low Prandtl number free convection, the neglected effects may well make themselves more strongly felt than for higher Prandtl number fluids.

The present note is concerned with the effects of transverse pressure gradient and streamwise second derivatives on the local heat transfer for laminar free convection on an isothermal vertical plate. The analysis is performed as a perturbation of the classical boundary-layer formulation. The separate effects of each of the aforementioned factors are studied as well as their simultaneous effects. Numerical calculations have been performed for Prandtl numbers of 0·03 and 0·003, which bound the liquid metal range, and also for $Pr = 0.733$ to demonstrate that the factors under consideration are negligible outside the liquid metal range. Grashof numbers delineating the threshold of significant departures from classical boundary-layer results are given.

Important contributions to the study of departures from classical free convection boundary-layer theory have been made by Yang *et al.* In one paper [1], consideration was given to the influence of the streamwise velocity induced in the ambient fluid by the transverse velocity of the classical boundary-layer solution. On the basis of computations for $Pr = 0.72$ and 10, it was found that the surface heat transfer was only very slightly affected, and then, in a manner just opposite to the trends of experiment. The present authors have chosen to by-pass the induced velocity effect in favour of those mentioned earlier, recognizing that this factor should ultimately be considered in a complete treatment of the problem.* In a second paper [2], Yang *et al.* investigated the range of very low Grashof numbers ($\leqslant 1$) by a perturbation of the pure conduction solution. Once again, calculations were performed for $Pr = 0{\cdot}72$ and 10.

ANALYSIS

Consideration is given to a semi-infinite vertical plate,† with x measuring distances vertically upward along the plate from the leading edge and y measuring distances normal to the plate. The wall and ambient temperatures are respectively denoted by T_w and T_∞. The starting point of the analysis is the following conservation equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} - \rho g + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (2)$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad (3)$$

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \quad (4)$$

Fluid properties are regarded as constant, except for the density appearing in the body force, which is expressed as

$$\rho = \rho_\infty - \beta\rho(T - T_\infty). \quad (5)$$

Viscous dissipation and compression work are neglected.

In contrast to the classical boundary-layer formulation, $\partial p/\partial x$ will not be evaluated in the ambient fluid since p is no longer being regarded as a function of x alone. Instead, the pressure is eliminated from the problem by taking $\partial/\partial y$ of equation (2), $\partial/\partial x$ of equation (3), and subtracting. This gives rise to a single, consolidated momentum equation of higher order. Next, a perturbation solution is sought in the form

$$\psi = 4\nu(Gr_x/4)^{\frac{1}{4}}\,[f_0(\eta) + \xi f_1(\eta) + \ldots] \quad (6)$$

$$\theta = (T - T_\infty)/(T_w - T_\infty) = \theta_0(\eta) + \xi\theta_1(\eta) + \ldots \quad (7)$$

where

$$\eta = \frac{y}{x}\left(\frac{Gr_x}{4}\right)^{\frac{1}{4}}, \qquad Gr_x = \frac{g\beta(T_w - T_\infty)x^3}{\nu^2}. \quad (8)$$

* The comparison is made for the isothermal vertical plate, with the boundary-layer thickness defined as the distance from the plate surface at which $(T - T_\infty)/(T_w - T_\infty) = 0{\cdot}02$.

† The influence of the induced velocity is of smaller order of magnitude for the semi-infinite vertical plate than for the plate of finite height.

The velocity components are derived from ψ by the usual differentiations: $\partial\psi/\partial y = u$, $\partial\psi/\partial x = -v$. The quantities $f_0(\eta)$ and $\theta_0(\eta)$ are the classical boundary-layer solutions. ξ is a stretching of x whose form is initially represented as cx^m.

Upon substituting equations (6) and (7) into the consolidated momentum equation and into the energy equation, one finds, after grouping terms multiplied by like powers of ξ (specifically, ξ^0 and ξ),*

$$f_0''' + 3f_0''f_0 - 2(f_0')^2 + \theta_0 = 0, \qquad \theta_0'' + 3Pr f_0\theta_0' = 0 \quad (9)$$

$$f_1''' + 3f_0f_1'' + 2f_0'f_1' - 3f_0''f_1 + \theta_1 = \tfrac{1}{8}\int_\eta^\infty F(\eta)\,d\eta \quad (10a)$$

$$\theta_1''/Pr + 3f_0\theta_1' + 6f_0'\theta_1 - 3f_1\theta_0' = \tfrac{1}{8}\,G(\eta). \quad (10b)$$

In addition, the appropriate form for ξ is found to be

$$\xi = \frac{1}{Gr_x^{\frac{1}{4}}}. \quad (11)$$

Equations (9) are the classical free convection boundary-layer equations and need not be discussed further here. The governing equations for the first order perturbations are expressed by equations (10). The forms of the F and G functions appearing on the right-hand sides of equations (10a) and (10b) depend on the specifics of the case being investigated. Consideration is given here to four cases involving the separate and simultaneous action of the transverse pressure gradient and streamwise second derivatives:

I. Transverse pressure gradient effect: $\partial^2u/\partial x^2$ and $\partial^2T/\partial x^2$ are suppressed in equations (2) and (4)

$$F_I = \eta^2(6f_0'f_0'' - \theta_0') + \eta[3f_0''' - 5(f_0')^2 + 3f_0f_0''] - 27f_0f_0' - 3f_0'' \quad (12a)$$

$$G_I = 0. \quad (12b)$$

II. $\partial^2u/\partial x^2$ effect: $\partial^2T/\partial x^2$ is suppressed in equation (4) and equation (3) is eliminated altogether so that $p = p(x)$.

$$F_{II} = \eta^2(-3f_0f_0''' + f_0'f_0'' - \theta_0') + \eta(3f_0'') - 3f_0'' \quad (13a)$$

$$G_{II} = 0. \quad (13b)$$

III. $\partial^2T/\partial x^2$ effect: $\partial^2u/\partial x^2$ is suppressed in equation (2) and equation (3) is eliminated altogether so that $p = p(x)$.

$$F_{III} = 0 \quad (14a)$$

$$G_{III} = -\frac{1}{Pr}(5\eta\theta_0' + \eta^2\theta_0'') \quad (14b)$$

IV. Simultaneous effects of transverse pressure gradient and streamwise second derivatives: all terms of equations (1–4) are retained.

$$F_{IV} = F_I + F_{II}, \qquad G_{IV} = G_{III}. \quad (15)$$

* To this order, the term $\partial^2v/\partial x^2$ does not contribute.

Owing to the linearity of equations (10), it is readily verified that case IV is a linear combination of cases I, II, and III. The boundary conditions appropriate to equations (10a) and (10b) are $f_1(0) = f'_1(0) = \theta_1(0) = f'_1(\infty) = \theta_1(\infty) = 0$.

The solutions of equations (9, 10a, 10b) were carried out numerically on an IBM 1130 computer at the Universidade do Brasil. It may be of interest to note that to achieve the desired accuracy, the solutions for $Pr = 0{\cdot}003$, 0·03, and 0·733 were respectively run to terminal η values of 180, 65, and 15.

RESULTS AND DISCUSSION

The quantity of primary interest in the present study is the local heat transfer per unit time and area as given by Fourier's law $q = -k(\partial T/\partial y)_{y=0}$. In terms of the variables of the analysis, q is expressible as

$$\frac{q}{q_0} = 1 + \frac{1}{Gr_x^{\frac{1}{4}}}\frac{\theta'_1(0)}{\theta'_0(0)} \tag{16}$$

where

$$q_0 = (k/x)(T_w - T_\infty)(Gr_x/4)^{\frac{1}{4}}[-\theta'_0(0)]. \tag{16a}$$

q_0 represents the local heat flux corresponding to the classical boundary-layer solution. Thus, the deviation of q/q_0 from unity is a direct measure of the effects of transverse pressure gradient and streamwise second derivatives. The magnitude of q/q_0 depends on the ratio $\theta'_1(0)/\theta'_0(0)$ and on the Grashof number Gr_x. Table 1 lists the θ' results in a manner appropriate to the evaluation of equation (16).

Considering Table 1 in conjunction with equation (16), it is seen that the combined effect of transverse pressure gradient and streamwise second derivatives (case IV) is to increase the local heat transfer, that is, $q/q_0 \geqslant 1$. The magnitude of the effect, as represented by the magnitude of $\theta'_1(0)/\theta'_0(0)$, increases greatly with decreasing Prandtl number (last column, Table 1). Further inspection of the table reveals that the dominant factor among those considered here is the longitudinal conduction $\partial^2T/\partial x^2$ (case III). The transverse pressure gradient (case I) also contributes to the augmentation of q, while the longitudinal shear $\partial^2u/\partial x^2$ (case II) tends to diminish q. For $Pr \sim 1$, the transverse pressure gradient and longitudinal shear contributions are essentially equal. With decreasing Pr, the former overshadows the latter, but is still secondary to the longitudinal conduction.

In view of the magnitudes of $\theta'_1(0)/\theta'_0(0)$ given in Table 1 (case IV), it is evident from equation (16) that for very low Prandtl numbers, Gr_x must be quite large in order that the classical boundary-layer results be applicable (i.e. q/q_0 close to unity). In this connection, it is illuminating to present (Table 2) the values of Gr_x at which $q/q_0 = 1{\cdot}05$, such a 5 per cent effect being regarded as the threshold of significant deviations from the classical boundary-layer theory. For values of Gr_x less than those of the table, $q/q_0 > 1{\cdot}05$. Therefore, with respect to the factors investigated here, the Gr_x of Table 2 represent lower bounds below which the classical boundary-layer theory cannot be applied without incurring significant error. It is evident that the range of validity of classical boundary-layer theory is significantly reduced at very low Prandtl numbers. Moreover, in light of Table 1, longitudinal conduction may be identified as the dominant factor causing the deviations from the classical theory.

Table 2. Values of Gr_x Corresponding to $q/q_0 = 1{\cdot}05$, Case IV

Pr	Gr_x
0·733	82
0·03	$2{\cdot}1 \times 10^4$
0·003	$1{\cdot}5 \times 10^6$

The foregoing finding that the local heat transfer exceeds that of classical boundary-layer theory at low Grashof numbers is in qualitative accord with experiment. However, quantitative comparison is not possible because the available experimental information pertains to overall rather than to local heat transfer [3]. Furthermore, there is very little experimental data for liquid metals, such fluids being the ones for which the low Grashof number effects are accentuated. For gases ($Pr \sim 0{\cdot}7$), measurements of *overall* heat transfer deviate from boundary-layer theory at overall Grashof numbers Gr_L of approximately 10^4 and below (L = plate length). By physical reasoning, it is readily

Table 1. Values of $\theta'_0(0)$ and $\theta'_1(0)/\theta'_0(0)$

Pr	$-\theta'_0(0)$	$\theta'_1(0)/\theta'_0(0)$			
		Case I	Case II	Case III	Case IV
0·733	0·507906	0·07778	−0·08995	0·46410	0·45192
0·03	0·134644	1·6145	−0·03871	5·7152	7·2910
0·003	0·0451774	17·474	−0·03893	44·791	62·227

understood that the overall heat transfer is more sensitive to deviations from boundary-layer theory than is the local heat transfer. Thus, the value of Gr_L at which the overall heat transfer begins to deviate from that of boundary-layer theory should be higher than the value of Gr_x at which deviations in the local heat transfer first occur. This reasoning is substantiated by comparing the value $Gr_x \sim 10^2$ from Table 2 ($Pr = 0{\cdot}733$) with the aforementioned value $Gr_L \sim 10^4$ from experiment.

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TURBULENT SUBLAYER TEMPERATURE DISTRIBUTION INCLUDING WALL INJECTION AND DISSIPATION

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NOMENCLATURE

C_f,	skin friction coefficient;
C_H,	Stanton number;
k,	conductivity;
Ec,	Eckert number, $Ec = u_\infty^2/[gc_p(T_w - T_\infty)]$;
Pr,	Prandtl number, $Pr = \mu c_p/k$;
Re,	Reynolds number, $Re = u_\infty x/\nu$;
t,	temperature fluctuation;
T,	temperature;
u, v, w,	velocity fluctuations;
U, V, W,	mean velocity;
u^*,	friction velocity, $u^* = \sqrt{(\tau_0/\rho)}$;
x, y, z,	distance co-ordinates.

Greek symbols

μ,	absolute viscosity;
ν,	kinematic viscosity;
ρ,	density;
τ,	shear stress.

INTRODUCTION

FOR HEAT and mass transport at Prandtl and Schmidt numbers different than one the transfer rates are governed by the mechanisms which dominate near the wall. Most analyses depend on arbitrary or semi-empirical expressions for the distribution of the eddy diffusivity coefficient near the wall which do not even obey the governing differential equations.

Expressions have been derived for the assymptotic forms of momentum and heat transport near the wall by Tien and Wasan [1] and Tien [2]; while the effect of wall transpiration on velocity and momentum transport has been considered by Meroney [3]. This note extends the argument to the case of the thermal boundary layer with transpiration.

ANALYSIS NEAR WALL

For uniform turbulent flow over an infinite two dimensional flat plate of uniform wall temperature, changes of the flow variables in the x-direction are negligible near the wall compared to changes in the y-direction. Hence, the equations of motion and energy for a constant property incompressible flow may be written as

$$V_y = 0 \tag{1}$$

$$VU_y = \nu U_{yy} - (\overline{uv})_y \tag{2}$$

$$\rho c_p V T_y = k T_{yy} - (\rho c_p \overline{vt})_y + \mu\phi \tag{3}$$

where ϕ is the dissipation function and may be expressed as

$$\phi = \left(\frac{\partial U}{\partial y}\right)^2 + \frac{\partial^2 \overline{v}^2}{\partial y^2} + \left(\overline{\frac{\partial u}{\partial x}}\right)^2 + \left(\overline{\frac{\partial w}{\partial z}}\right)^2 + \left(\overline{\frac{\partial v}{\partial y}}\right)^2 + \left(\overline{\frac{\partial u}{\partial y}}\right)^2 + \left(\overline{\frac{\partial v}{\partial z}}\right)^2 + \left(\overline{\frac{\partial w}{\partial x}}\right)^2 + \left(\overline{\frac{\partial w}{\partial y}}\right)^2.$$

Near the wall the velocity and temperature terms (T, t, u,

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